# ECED 3300
# Tutorial 4

## Problem 1

*Show that*

$$\nabla \times \left(\frac{\mathbf{B} \times \mathbf{r}}{2}\right) = \mathbf{B},$$

*where* $\mathbf{r} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z$ *and* $\mathbf{B}$ *is a constant vector.*

### Solution

By definition,

$$\mathbf{B} \times \mathbf{r} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ B_x & B_y & B_z \\ x & y & z \end{vmatrix} = (zB_y - yB_z)\mathbf{a}_x + (xA_z - zA_x)\mathbf{a}_y + (yB_x - xB_y)\mathbf{a}_z.$$

It then follows that

$$\nabla \times \left(\frac{\mathbf{B} \times \mathbf{r}}{2}\right) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \partial_x & \partial_y & \partial_z \\ \frac{1}{2}(zB_y - yB_z) & \frac{1}{2}xA_z - zA_x & \frac{1}{2}(yB_x - xB_y) \end{vmatrix},$$

Thus,

$$\nabla \times \left(\frac{\mathbf{B} \times \mathbf{r}}{2}\right) = \mathbf{a}_x(B_x/2 + B_x/2) - \mathbf{a}_y(-B_y/2 - B_y/2) + \mathbf{a}_z[B_z/2 - (-B_z/2)] = \mathbf{B}.$$

Q.E.D. The physical significance of the result is as follows. We will learn that a magnetic flux density $\mathbf{B}$ is related to a vector potential $\mathbf{A}$ via a curl, $\mathbf{B} = \nabla \times \mathbf{A}$. Hence the vector potential of a uniform magnetic field is $\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$.

## Problem 2

*Show that for any vector field,* $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

### Solution

Introduce an auxiliary vector field, $\mathbf{F} = \nabla \times \mathbf{A}$. By the definition of divergence, $\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$. Using the components of a cirl in the Cartesian coordinates for the Formula Sheet, we can rewrite our divergence as

$$\nabla \cdot \mathbf{F} = \partial_x \partial_y A_z - \partial_z A_y + \partial_y \partial_z A_x - \partial_x A_z + \partial_z \partial_x A_y - \partial_y A_x.$$

Multiplying through and taking the partials, gives

$$\nabla \cdot (\nabla \times \mathbf{A}) = \partial^2_{xy} A_z - \partial^2_{xz} A_y + \partial^2_{yz} A_x - \partial^2_{yx} A_z + \partial^2_{zx} A_y - \partial^2_{zy} A_x = 0.$$

This follows by noticing that mixed partial derivatives are the same, $\partial^2_{xy} A_z = \partial^2_{yx} A_z$ etc. Q.E.D. *Note: This vector calculus identity enables one to automatically satisfy the divergence equation for the magnetic flux density by introducing the vector potential.*

# Problem 3

*Verify Stokes's theorem, $\oint_C d\mathbf{l} \cdot \mathbf{F} = \int d\mathbf{S} \cdot (\nabla \times \mathbf{F})$, for the vector field $\mathbf{F} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$. Here $C$ a circle of radius $R$ centered at the origin in the $xy$-plane and $S$ is curved and upper surfaces of the cylinder of radius $R$ and height $h$ rimmed by the circle.*

**Solution**

1) On the one hand, $\mathbf{F}$ can be converted to the cylindrical coordinates on $C$ using the Formula Sheet to yield,

$$\mathbf{F} = \rho \sin\phi(\mathbf{a}_\rho \cos\phi - \mathbf{a}_\phi \sin\phi) - \rho\cos\phi(\mathbf{a}_\rho \sin\phi + \mathbf{a}_\phi \cos\phi) + \mathbf{a}_z z = -\mathbf{a}_\phi \rho + \mathbf{a}_z z.$$

Further, on $C$, $d\mathbf{l} = \mathbf{a}_\phi R d\phi$ and $\rho = R$ and $z = 0$ implying that.

$$\mathbf{F} = -\mathbf{a}_\phi R.$$

It then follows that

$$\oint_C d\mathbf{l} \cdot \mathbf{F} = -R^2 \int_0^{2\pi} d\phi\, (\mathbf{a}_\phi \cdot \mathbf{a}_\phi) = -2\pi R^2.$$

2) On the other hand,

$$\nabla \times \mathbf{F} = \frac{1}{\rho} \begin{vmatrix} \mathbf{a}_\rho & \rho\mathbf{a}_\phi & \mathbf{a}_z \\ \partial_\rho & \partial_\phi & \partial_z \\ 0 & -\rho^2 & z \end{vmatrix} = \frac{\mathbf{a}_z}{\rho}(-2\rho) = -2\mathbf{a}_z.$$

Thus, only flux through the top is nonzero. In this case, $d\mathbf{S} = \mathbf{a}_z \rho d\rho d\phi$. Hence,

$$\int d\mathbf{S} \cdot (\nabla \times \mathbf{F}) = \int_0^{2\pi} d\phi \int_0^R d\rho\, \rho(-2\mathbf{a}_z \cdot \mathbf{a}_z) = -2\pi R^2.$$

# Problem 4

*Verify Stokes's theorem for the vector field* $\mathbf{B} = \rho\cos\phi\mathbf{a}_\rho + \sin\phi\mathbf{a}_\phi$ *by evaluating*
*(a)* $\oint_C d\mathbf{l}\cdot\mathbf{B}$ *over a semicircular contour of radius 2 centered at the origin in the upper half-plane* $xy$*.*
*(b)* $\int_S d\mathbf{S}\cdot(\nabla\times\mathbf{B})$ *over the surface of the semicircle.*

**Solution**

(a) The path consists of two parts: $-2 \le x \le 2$, where $d\mathbf{l} = \mathbf{a}_x dx$, and $\rho = 2$, where $d\mathbf{l} = 2\mathbf{a}_\phi d\phi$. In the interval $-2 \le x \le 2$, $\phi = 0$ for positive $x$ and $\phi = \pi$ for negative $x$, implying that in Cartesian coordinates the field is given by

$$\mathbf{B} = \begin{cases} x\mathbf{a}_x, & x > 0 \\ -x\mathbf{a}_x, & x < 0 \end{cases}$$

The field along the arc $\rho = 2$ can be represented in polar coordinates as $\mathbf{B} = 2\cos\phi\mathbf{a}_\rho + \sin\phi\mathbf{a}_\phi$. It then follows that

$$\oint_C d\mathbf{l}\cdot\mathbf{B} = -\int_{-2}^{0} dx\, x + \int_0^2 dx\, x + 2\int_0^\pi d\phi\sin\phi = 4 + 2\cos\phi|_\pi^0 = 8.$$

(b) On the other hand,

$$\nabla\times\mathbf{B} = \frac{\sin\phi}{\rho}(1+\rho)\mathbf{a}_z.$$

It then follows that

$$\int_S d\mathbf{S}\cdot(\nabla\times\mathbf{B}) = \int_0^2 d\rho\rho\int_0^\pi d\phi\frac{\sin\phi}{\rho}(1+\rho) = \int_0^\pi d\phi\sin\phi\int_0^2 d\rho(1+\rho) = (\rho + \tfrac{1}{2}\rho^2)\Big|_0^2 \times \cos\phi|_\pi^0 = 8.$$

# Problem 5

*Show that the area* $A$ *enclosed by a curve* $C$ *lying entirely in the* $xy$*-plane is given by the magnitude of the vector* $\mathbf{F}$*,*

$$\mathbf{F} = \tfrac{1}{2}\oint_C \boldsymbol{\rho}\times d\mathbf{l},$$

*where* $\boldsymbol{\rho} = \mathbf{a}_x x + \mathbf{a}_y y$*.*

**Solution**

It follows from the definition that $\mathbf{F} = |\mathbf{F}|\mathbf{a}_z$. Hence,

$$|\mathbf{F}| = \mathbf{F} \cdot \mathbf{a}_z = \tfrac{1}{2} \oint_C \mathbf{a}_z \cdot (\boldsymbol{\rho} \times d\mathbf{l}) = \tfrac{1}{2} \oint_C d\mathbf{l} \cdot (\mathbf{a}_z \times \boldsymbol{\rho}),$$

Applying the Stokes theorem to the right-hand side, we obtain,

$$|\mathbf{F}| = \tfrac{1}{2} \oint_C d\mathbf{l} \cdot (\mathbf{a}_z \times \boldsymbol{\rho}) = \tfrac{1}{2} \int_S d\mathbf{S} \cdot \nabla \times (\mathbf{a}_z \times \boldsymbol{\rho}),$$

where for a closed curve in the $xy$-plane

$$d\mathbf{S} = \mathbf{a}_z dS.$$

Notice that

$$\mathbf{a}_z \times \boldsymbol{\rho} = x(\mathbf{a}_z \times \mathbf{a}_x) + y(\mathbf{a}_z \times \mathbf{a}_y) = x\mathbf{a}_y - y\mathbf{a}_x.$$

Further,

$$\nabla \times (\mathbf{a}_z \times \boldsymbol{\rho}) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = 2\mathbf{a}_z.$$

Therefore,

$$|\mathbf{F}| = \tfrac{1}{2} \times 2 \int_S dS(\mathbf{a}_z \cdot \mathbf{a}_z) = \int_S dS = A.$$

Q.E.D.